CENTRE DE PHYSIQUE THÉORIQUE
CNRS - Luminy, Case 907
13288 Marseille Cedex 9

NONCOMMUTATIVE YANG-MILLS AND NONCOMMUTATIVE RELATIVITY: A BRIDGE OVER TROUBLE WATER

Lionel CARMINATI [1]
Bruno IOCHUM [1]
Thomas SCHÜCKER [1]

Abstract

Connes' view at Yang-Mills theories is reviewed with special emphasis on the gauge invariant scalar product. This landscape is shown to contain Chamseddine and Connes' noncommutative extension of general relativity restricted to flat space-time, if the top mass is between 172 and 204 GeV. Then the Higgs mass is between 188 and 201 GeV.

PACS-92: 11.15 Gauge field theories
MSC-91: 81T13 Yang-Mills and other gauge theories

june 1997

CPT-96/P.3477
hep-th/yymmxxx

[1] and Université de Provence
carminati@cpt.univ-mrs.fr iochum@cpt.univ-mrs.fr schucker@cpt.univ-mrs.fr

1 Introduction

Einstein was a passionate sailor. We speculate that this was no accident. The subtle harmony between geometries and forces becomes palpable to the sailor, he sees the curvature of the sail and feels the force that it produces. Before Einstein, it was generally admitted that forces are vector fields in an Euclidean space, $\mathbb{R}^3$, the scalar product being necessary to define work and energy. Einstein generalized Euclidean to Minkowskian and Riemannian geometry and we have two *dreisätze* or *règles de trois*. Take Coulomb's static law for the electric field with coupling constant ϵ_0 and add Minkowskian geometry with its scale c, the speed of light: you obtain Maxwell's theory. In particular, there appears the magnetic field with feeble coupling constant $\mu_0 = 1/(c^2\epsilon_0)$. Maxwell's theory is celebrated today as Abelian or should we say, commutative Yang-Mills theory. The second *dreisatz* starts from Newton's (static) universal law of gravitation, adds Riemannian geometry to obtain general relativity with new feeble, gravito-magnetic forces.

Connes proposes two more dreisätze. Take a certain Yang-Mills theory with coupling constant g, coupled to a Dirac spinor of mass m. Add noncommutative geometry [1] with an energy scale Λ: you obtain a Yang-Mills-Higgs theory [2],[3]. The symmetry breaking scalar becomes a magnetic field of the Yang-Mills field and its mass and self-coupling λ are constrained in terms of g, m and Λ. His second dreisatz starts from general relativity, adds noncommutative geometry to obtain the Einstein-Hilbert action plus the Yang-Mills-Higgs action [4][5]. Now the Yang-Mills and the Higgs fields are magnetic fields of the gravitational field. Again there are constraints on λ, but they are different.

Let us call noncommutative Yang-Mills the third and noncommutative relativity the fourth dreisatz. Note however that — unlike with supersymmetry — you cannot take any Yang-Mills theory and put 'noncommutative' in front [6][7][8]. Note also that, behind noncommutative relativity, there stands a genuine noncommutative extension of Einstein's principle of general relativity, the spectral principle.

One of the attractive features of noncommutative geometry is to unify gauge couplings with scalar self-couplings and Yukawa couplings. These couplings will be at the center of our discussion. They are related to the set of all scalar products on a given space,which is a cone: in the case of gauge invariant scalar products on the Lie algebra of a Yang-Mills theory, the gauge couplings are positive coordinates on this cone. For us, a noncommutative geometry is given by a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. The motivation for this noncommutative geometry comes from Connes' theorem [4] establishing a one-to-one correspondence between *commutative* spectral triples and Riemannian spin geometries [9]. $\mathcal{A}$ is an associative involution algebra with unit, represented faithfully on a Hilbert space $\mathcal{H}$ on which also the self-adjoint 'Dirac' operator $\mathcal{D}$ acts. Physically, the Hilbert space is spanned by the fermions. The Dirac operator serves several purposes: it defines the kinetic term for fermions, it allows to construct differential forms — the Yang-Mills fields are 1-forms —, it is used as ultra-violet regulator to define the

scalar product and above all, it defines the metric structure of space-time.

So, in this paper, playing with the scalar products and the experimental accuracies of the gauge couplings and the masses, we consider the noncommutative Yang-Mills and noncommutative relativity theories as effective theories, using the renormalization flow and show that they can be related according to certain constraints on the Higgs and top masses.

2 Noncommutative relativity

The dynamical variable of gravity is the metric on space-time. Einstein used the matrix $g^{\mu\nu}(x)$ of the metric g with respect to a coordinate system x^μ to parameterize the set of all metrics on a fixed space-time M. The coordinate system being unphysical, Einstein required his field equations for the metric to be covariant under coordinate transformations, the principle of general relativity. Elie Cartan used tetrads, *repères mobiles*, to parameterize the set of all metrics. This parameterization allows to generalize the Dirac operator $\mathcal{D}$ to curved space-times and also reformulates general relativity as a gauge theory under the Lorentz group. Connes [4] goes one step further by relating the set of all metrics to the set of all Dirac operators. The Einstein-Hilbert action, from this point of view, is the Wodzicki residue of the second inverse power of the Dirac operator [10] and is computed most conveniently from the second coefficient of the heat kernel expansion of the Dirac operator squared. The heat kernel expansion [11] is an old friend [12] from quantum field theory in curved space-time, from its formal relation to the one-loop effective action

$$S_{\mathit{eff}} = \mathrm{tr}\,\log(\mathcal{D}^2/\Lambda^2), \qquad \Lambda \text{ a cutoff.} \tag{1}$$

This relation has been used by Sakharov [13] to induce gravity from quantum fluctuations, leading however to a negative Newton constant [14].

By generalizing the metric, the Dirac operator plays a fundamental role in noncommutative geometry. To describe Yang-Mills theories, Connes considers the tensor product of space-time and internal space in this new geometry, a natural point of view because the fermionic mass matrix qualifies as Dirac operator on internal space. This cheap tensor product unifies space-time diffeomorphisms with internal gauge transformations by extending Einstein's principle of general relativity to noncommutative geometry. Remember that Einstein constructed general relativity in two steps, by applying his principle first to matter, then to the gravitational field itself. Connes follows this pattern and of course in his case, the spinors are the matter.

To generalize the Dirac operator from flat to curved space-time (locally), it is sufficient to write the Dirac operator first in flat space-time but with respect to noninertial coordinates. A straightforward calculation produces the *covariant* Dirac operator that contains the spin connection ω. Although of vanishing curvature, ω contains already a lot of physics, e.g. the centrifugal and Coriolis accelerations in the coordinates of the rotating disk, the quantum interference pattern of neutrons [15] in oscillating coordinates. Then, the generalization to curved

space is easy where ω describes the (minimal) coupling of the spinor to the gravitational field. In Einstein's spirit, the covariant Dirac operator is obtained by acting with the diffeomorphism group on the flat Dirac operator. But the diffeomorphism group is just the automorphism group of the associative (and commutative) algebra $\mathcal{C}^\infty(M)$ representing space-time in noncommutative geometry. On the other hand, the product of space-time and internal space is represented in this geometry by the tensor product of $\mathcal{C}^\infty(M)$ with a matrix algebra. Its automorphism group is the semi-direct product of the diffeomorphisms and unitaries of the matrix algebra, the internal gauge transformations. The diffeomorphisms are the outer, the gauge transformations are the inner automorphisms. And what do we get when this entire automorphism group acts on the flat Dirac operator? We get the total covariant Dirac operator containing the spin connection, the gauge connection and the Higgs [3]. In other words, we get the minimal couplings of the Dirac spinor to the gravitational and Yang-Mills fields and its Yukawa couplings to the Higgs field. In Connes' words, the Higgs and Yang-Mills fields are noncommutative fluctuations of the metric. (Abelian Yang-Mills theories do not have such fluctuations.) Accordingly, Connes generalizes Einstein's principle of general relativity by postulating that only the intrinsic properties of the covariant Dirac operator be relevant for physics. Here intrinsic means invariant under automorphisms. Thus, these properties must concern the spectrum only. Spectral principle is a convenient name for Connes' generalization of the principle of general relativity.

So far, we have only the kinematic of the metric (and its fluctuations). To get its dynamics, Einstein developed the full power of the principle of general relativity and derived the Einstein-Hilbert action. In short this story: the $1/r^2$ in Newton's universal law is the Green function of the divergence, an operator of first order in the forces. We already know that the forces are encoded in the connection ω. Riemannian geometry tells us that the connection is obtained from first order derivatives of the metric. Therefore Einstein looked for a second order differential equation for the metric. The covariance under change of coordinates fixes this equation up to the cosmological constant to be the Einstein equation. Chamseddine & Connes [5] reedit this story using the spectral principle. It is stronger than Einstein's principle in the sense that for the metric only, the Einstein-Hilbert action follows without the use of Newton's law. In addition, the spectral principle fixes the action of the fluctuations to be the Yang-Mills action, the covariant Klein-Gordon action and the symmetry breaking Higgs potential. Warning: following physicists' habits, we have confused diffeomorphisms and coordinate transformations. Cleaning up this point leads to deep mathematics [16] and probably a further unification of general relativity and Yang-Mills theory: the reduction of the diffeomorphism group to an isometry group might take the form of a spontaneous symmetry break down.

2.1 The stiff action

In even dimensions, the spectrum of the Dirac operator is even and it is sufficient to consider the positive part of the spectrum which in the Euclidean is conveniently characterized by a distribution function

$$S = \mathrm{tr}\, f(\mathcal{D}^2/\Lambda^2), \tag{2}$$

where Λ is an energy cutoff and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a positive, smooth function with finite, strictly positive first 'momenta',

$$f_0 := \int_0^\infty u f(u)\,\mathrm{d}u, \quad f_2 := \int_0^\infty f(u)\,\mathrm{d}u, \quad f_4 := f(0). \tag{3}$$

If instead, f was the logarithm, this trace, after a proper renormalization, would be Sakharov's induced gravity action. The positive function f is universal: the action S can be computed asymptotically [17], that is up to terms of the order of Λ^{-2}, using the Lichnérowicz formula and the heat kernel expansion. The action depends only on the three momenta $f_0,\ f_2,\ f_4$ and takes the form:

$$\begin{aligned}\mathrm{tr}\, f(\mathcal{D}_{t,\mathrm{cov}}^2/\Lambda^2) \approx \int_M [&-\tfrac{1}{16\pi} m_P(\Lambda)^2\, R + \Lambda_C(\Lambda) \\ &+ \tfrac12 g_3(\Lambda)^{-2}\,\mathrm{tr}\, F^{(3)}_{\mu\nu} F^{(3)\mu\nu} + \tfrac12 g_2(\Lambda)^{-2}\,\mathrm{tr}\, F^{(2)}_{\mu\nu} F^{(2)\mu\nu} + \tfrac14 g_1(\Lambda)^{-2}\, F^{(1)}_{\mu\nu} F^{(1)\mu\nu} \\ &+ \tfrac12 (\mathrm{D}_\mu\varphi)^* \mathrm{D}^\mu\varphi + \lambda(\Lambda)|\varphi|^4 - \tfrac12 \mu(\Lambda)^2 |\varphi|^2 \\ &- a(\Lambda) C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \tfrac{1}{12} |\varphi|^2 R \quad]\,(\det g_{\cdot\cdot})^{1/2}\,\mathrm{d}^4x.\end{aligned} \tag{4}$$

Here $\mathcal{D}_{t,\mathrm{cov}}$ is the total, covariant Dirac operator of the standard model of electroweak and strong interactions with $N = 3$ generations of quarks and leptons. It follows that φ is an isospin doublet. After a proper normalization of the kinetic terms and a shift of the Higgs field by its vacuum expectation value, $|\varphi| = v(\Lambda) = \mu(\Lambda)/(2\sqrt{\lambda}(\Lambda))$, we can identify Newton's constant $G = 16\pi\hbar c\, m_P^2$, the cosmological constant Λ_C and the other coupling constants

$$m_P(\Lambda)^2 = \tfrac{1}{3\pi} f_2 \left[15N - 2\frac{L^2}{Q}\right] \Lambda^2, \tag{5}$$

$$L(\Lambda) := 3(m_t^2 + m_c^2 + m_u^2 + m_b^2 + m_s^2 + m_d^2) + m_\tau^2 + m_\mu^2 + m_e^2, \tag{6}$$

$$Q(\Lambda) := 3(m_t^4 + m_c^4 + m_u^4 + m_b^4 + m_s^4 + m_d^4) + m_\tau^4 + m_\mu^4 + m_e^4, \tag{7}$$

$$\Lambda_C(\Lambda) = \tfrac{1}{4\pi^2}\left[15N f_0 - \frac{f_2^2}{f_4}\frac{L^2}{Q}\right]\Lambda^4, \tag{8}$$

$$g_3(\Lambda)^{-2} = \tfrac{N}{3\pi^2} f_4, \tag{9}$$

$$g_2(\Lambda)^{-2} = \tfrac{N}{3\pi^2} f_4, \tag{10}$$

$$g_1(\Lambda)^{-2} = \tfrac53 \tfrac{N}{3\pi^2} f_4, \tag{11}$$

$$\lambda(\Lambda)^{-1} = \tfrac{1}{\pi^2}\, f_4\, \frac{L(\Lambda)^2}{Q(\Lambda)} = \tfrac{3}{\pi^2}\, f_4 (1 + 2\frac{m_b(\Lambda)^2}{m_t(\Lambda)^2} + O(\frac{m_\tau(\Lambda)^2}{m_t(\Lambda)^2})), \tag{12}$$

$$\mu(\Lambda)^2 = 2\frac{f_2}{f_4}\Lambda^2, \tag{13}$$

$$a(\Lambda) = \tfrac{3N}{64\pi^2}\, f_4. \tag{14}$$

From now on, we ignore the gravitational part because we want to use the renormalization flow of the coupling constants and also because we want to compare this theory with the noncommutative Yang-Mills that, by the way, automatically has a vanishing cosmological constant as we shall see.

The constraints for the gauge couplings,

$$g_3(\Lambda) = g_2(\Lambda) \text{ and } \sin^2\theta_w(\Lambda) = \frac{g_2^{-2}}{g_1^{-2} + g_2^{-2}} = \frac{3}{8}$$

(9-11), are familiar from grand unification and force us to assume the big desert. Consequently all numerical considerations will be qualitative only. Indeed, the three gauge couplings $g_i(\Lambda)$, once fixed at the Z-mass to their experimental values $\sin^2\theta_w(m_Z) = 0.2315 \pm 0.0005$, see appendix, do not meet in a point anymore as was the case in the $SU(5)$ days. Today they define a triangle with $\Lambda = 10^{13} - 10^{17}$ GeV and $\sqrt{\frac{5}{3}}g_1(\Lambda), g_2(\Lambda), g_3(\Lambda)$ are in the interval $0.52 - 0.56$, Figure 1. Details on the renormalization group flow can be found in the appendix. For the noncommutative constraints (9-11), this means that f_4 cannot take a precise value, $\frac{1}{4\pi^2}\, f_4 = 0.80 - 0.94$.

2.2 The soft Einstein-Hilbert action

Of course, we may try to do better by introducing more parameters. Let z', the 'noncommutative coupling constant', be a positive operator on the fermionic Hilbert space that commutes with the representation and the Dirac operator. For the standard model, this z' contains four positive numbers x', y_1', y_2', y_N'. We soften the action (4) to $\mathrm{tr}\,[z' f(\mathcal{D}^2_{t,cov}/\Lambda^2)]$. Then the constraints read [18]:

$$g_3(\Lambda)^{-2} = \tfrac{1}{9\pi^2} f_4 N x', \tag{15}$$

$$g_2(\Lambda)^{-2} = \tfrac{1}{12\pi^2} f_4 (N x' + y_1' + y_2' + y_N'), \tag{16}$$

$$g_1(\Lambda)^{-2} = \tfrac{1}{12\pi^2} f_4 (\tfrac{11}{9} N x' + 3(y_1' + y_2' + y_N')), \tag{17}$$

$$\lambda(\Lambda)^{-1} = \tfrac{1}{\pi^2} f_4 \frac{L(\Lambda)^2}{Q(\Lambda)} \tag{18}$$

$$L(\Lambda) = x'(m_t^2 + m_c^2 + m_u^2 + m_b^2 + m_s^2 + m_d^2) + y_3' m_\tau^2 + y_2' m_\mu^2 + y_1' m_e^2, \tag{19}$$

$$Q(\Lambda) = x'(m_t^4 + m_c^4 + m_u^4 + m_b^4 + m_s^4 + m_d^4) + y_3' m_\tau^4 + y_2' m_\mu^4 + y_1' m_e^4, \tag{20}$$

$$\mu(\Lambda)^2 = 2\,\frac{f_2}{f_4}\,\Lambda^2 \tag{21}$$

If $z' = 1_{90}$, then $x' = 3,\ y_1' = y_2' = y_N' = 1$ and we recover the stiff relations (11-13).

2.3 The dominating top approximation and renormalization flow

The soft relations do not have the problem $g_3(\Lambda) = g_2(\Lambda)$ anymore, but we still cannot avoid the desert. In fact now

$$sin^2\theta_w(\Lambda) = \frac{N x' + (y_1' + y_2' + y_N')}{\frac{20}{9} N x' + 4(y_1' + y_2' + y_N')},$$

and the weak mixing angle is constrained for all Λ: $\frac{1}{4} < \sin^2\theta_w(\Lambda) < \frac{9}{20}$. From now on, we neglect all fermion masses with respect to the top mass. This approximation induces relative errors of the order of $m_b^2/m_t^2 = 0.0006$ and it reduces the number of positive parameters from four, x', y'_1, y'_2, y'_N to two, x' and $y' := y'_1 + y'_2 + y'_N$. In the one loop approximation, the evolution of the gauge couplings (54-56) decouples from the other couplings and we can solve the constraints (15-17) such that at the Z mass, they reproduce precisely the experimental values. The last non-empty constraint (18), $\lambda(\Lambda) = \frac{N}{9} g_3(\Lambda)^2$ then fixes the Higgs mass. With these approximations, we obtain:

- In the stiff case, $x' = y' = 3$, the incertainty on the cutoff is large:

$$\left.\begin{aligned} &\Lambda = (10^{13} - 10^{17})\ \text{GeV}, \\ &\tfrac{1}{12\pi^2} f_4 x' = \tfrac{1}{12\pi^2} f_4 y' = 0.80 - 0.94, \\ &m_H = 182 \pm 10 \pm 2 \pm 7\ \text{GeV}. \end{aligned}\right\}\ \text{stiff EH}$$

The first error is from the uncertainty in the noncommutative scale Λ, the second from the present experimental uncertainty in the gauge couplings, $g_3 = 1.218 \pm 0.026$, and the third from the uncertainty in the top mass, $m_t = 175 \pm 6$ GeV.

- In the soft case, the cutoff is sharp:

$$\left.\begin{aligned} &\Lambda = 0.96 \cdot 10^{10}\ \text{GeV}, \\ &\tfrac{1}{12\pi^2} f_4 x' = 0.578, \\ &\tfrac{1}{12\pi^2} f_4 y' = 1.369 \\ &m_H = 190 \pm 0 \pm 1 \pm 4\ \text{GeV}. \end{aligned}\right\}\ \text{soft EH}$$

Let us anticipate that this comparison will work quantitatively only for the stiff case. So if we spell out the soft case here, then not because we believe that it makes sense to fit numbers through the big desert with the indicated precision. Our aim is to assess the stability of the Higgs mass prediction and also to make the comparison with the noncommutative Yang-Mills easier.

3 The noncommutative Yang-Mills action

After this quick review of the noncommutative version of general relativity in flat space-time, we now turn to our main concern, noncommutative Yang-Mills theory, Connes' first dreisatz. The point will be that Connes' second dreisatz adds insight to the older one.

3.1 The conventional scalar products

To construct a Yang-Mills action $\int \mathrm{tr}\, F{*}F$, we need four ingredients: differential forms on space-time M, a Lie group G, 'the internal space', a scalar product on the space of differential forms

ΩM and an invariant scalar product on the Lie algebra $\mathfrak{g}$ of the group G. The gauge field A is a 1-form with values in $\mathfrak{g}$, its field strength or curvature is the 2-form $F := \mathrm{d}A + \frac{1}{2}[A, A]$ again with values in $\mathfrak{g}$. To construct the action which is a real number, we take the scalar products of the field strength with itself. The first scalar product involves the space-time metric g hidden in the Hodge star $*$, $(\omega, \kappa) := \int_M \omega^* * \kappa$, ω and κ differential forms of same degree. In components, e.g. for 2-forms $\omega = \frac{1}{2}\omega_{\mu\nu}\mathrm{d}x_\mu \mathrm{d}x^\nu$, we have $(\omega, \kappa) = \frac{1}{2}\int_M \omega^*_{\mu\nu}\kappa_{\mu'\nu'}g^{\mu\mu'}g^{\nu\nu'}(\det g_{\cdot\cdot})^{1/2}\,\mathrm{d}^4x$. We suppose M Euclidean, otherwise this scalar product would only be a pseudo scalar product. The second scalar product is on the Lie algebra. It only exists if the Lie group is compact. E.g. for $G = SU(n)$, the general invariant scalar product is $(a, b) = \frac{2}{g_n^2}\mathrm{tr}\,(a^*b)$, $a, b \in su(n)$ and the coupling constant g_n is a positive number. In general, the space of all scalar products is a cone whose coordinates are the coupling constants.

3.2 The axioms

Noncommutative geometry does to space-time M, a Riemannian manifold, what quantum mechanics did to phase space. An uncertainty relation is introduced by allowing the commutative algebra of functions $\mathcal{C}^\infty(M)$ to become noncommutative. Let us call $\mathcal{A}$ this new algebra that we still suppose real, associative and equipped with a unit and an involution. On phase space, $\mathcal{A}$ was just the algebra of observables. Now we want to define a distance on this new space that has lost its points. Following Connes, we need a faithful representation ρ of $\mathcal{A}$ via bounded operators on a Hilbert space $\mathcal{H}$, the space of fermions, and a selfadjoint 'Dirac' operator $\mathcal{D}$ on $\mathcal{H}$. Connes calls these three ingredients a spectral triple, $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. They satisfy axioms that are simply taken from the properties of the commutative case, $\mathcal{A} = \mathcal{C}^\infty(M)$, the Hilbert space $\mathcal{H}$ is the space of ordinary, square integrable Dirac spinors. An element f of $\mathcal{A}$ is a differentiable function on space-time, $f(x)$, and it acts on a spinor $\psi(x)$ by multiplication $(\rho(f)\psi)(x) := f(x)\psi(x)$. $\mathcal{D} = \partial\!\!\!/$ is the ordinary Dirac operator. Only recently Connes has completed the list of axioms [4] as to have a one-to-one correspondence between commutative spectral triples and Riemannian spin manifolds. To this end, he needed two other old friends from particle physics, a chirality operator χ and a real structure J. The chirality is a unitary operator of square one that commutes with the representation. Therefore χ decomposes the representation space into a left-handed piece $\frac{1-\chi}{2}\mathcal{H}$ and a right-handed piece $\frac{1+\chi}{2}\mathcal{H}$. In the commutative case, of course $\chi = \gamma_5$. The real structure is an anti-unitary operator that in the commutative case reduces to the charge conjugation operator C. J is of square plus or minus one, depending on space-time dimension and signature. Also depending on space-time dimension and signature, J commutes or anticommutes with χ. The charge conjugation as well decomposes the representation space into two pieces, particles and anti-particles, all together

$$\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_L^c \oplus \mathcal{H}_R^c. \tag{22}$$

Here are a few more properties from the commutative case that become axioms

- $\rho(a)$ commutes with $J\rho(\tilde{a})J^{-1}$, for all $a, \tilde{a} \in \mathcal{A}$,
- $\mathcal{D}\chi = -\chi\mathcal{D}$,
- $\mathcal{D}J = +J\mathcal{D}$,
- $[\mathcal{D}, \rho(a)]$ is bounded for all a in $\mathcal{A}$,
- $[\mathcal{D}, \rho(a)]$ commutes with $J\rho(\tilde{a})J^{-1}$, for all $a, \tilde{a}$ in $\mathcal{A}$.

The last axiom is called first order, because in the commutative case, it just says that the Dirac operator is a first order differential operator. The dimensionality of M can be recovered from the spectrum of the Dirac operator. Indeed for compact manifolds, the spectrum is discrete and the ordered eigenvalues λ_n grow like $n^{1/\dim M}$. This motivates the name spectral triple. Let us mention two more axioms. The orientability axiom relates the chirality to the volume form, a differential form of maximal degree. The Poincaré duality on manifolds is promoted to an axiom in quite an abstract form. We anticipate that, in the case of the standard model, this Poincaré duality will prohibit right-handed neutrinos [3].

3.3 Differential forms

Our next aim is to construct differential forms starting from a spectral triple. In the commutative case, we want this construction to reproduce de Rham's differential forms, ΩM.

We start with an auxiliary differential algebra $\Omega\mathcal{A}$, the universal differential envelope of $\mathcal{A}$: $\Omega^0\mathcal{A} := \mathcal{A}$. $\Omega^1\mathcal{A}$ is generated by symbols δa, $a \in \mathcal{A}$ with relations $\delta 1 = 0$, $\delta(aa') = (\delta a)a' + a\delta a'$. $\Omega^1\mathcal{A}$ consists of finite sums of terms of the form $a_0\delta a_1$, and likewise for higher degree p,

$$\Omega^p\mathcal{A} = \left\{ \sum_j a_0^j \delta a_1^j ... \delta a_p^j, \quad a_q^j \in \mathcal{A} \right\}.$$

The differential δ is defined by

$$\delta(a_0\delta a_1 ... \delta a_p) := \delta a_0 \delta a_1 ... \delta a_p.$$

The involution * is extended from the algebra $\mathcal{A}$ to $\Omega^1\mathcal{A}$ by putting $(\delta a)^* := \delta(a^*) =: \delta a^*$ and to the entire differential envelope by $(\varphi\psi)^* = \psi^*\varphi^*$. The next step is to extend the representation ρ from the algebra $\mathcal{A}$ to its envelope $\Omega\mathcal{A}$. This extension deserves a new name:

$$\pi : \Omega\mathcal{A} \longrightarrow \bigoplus_p \mathrm{End}(\mathcal{H}),$$

$$\pi(a_0\delta a_1 ... \delta a_p) := (-i)^p \rho(a_0)[\mathcal{D}, \rho(a_1)]...[\mathcal{D}, \rho(a_p)].$$

π is a representation of $\Omega\mathcal{A}$ as graded involution algebra, and we are tempted to define also a differential, again denoted by δ, on $\pi(\Omega\mathcal{A})$ by $\delta\pi(\hat{\varphi}) := \pi(\delta\hat{\varphi})$. However, this definition does

not make sense because there are forms $\hat\varphi \in \Omega\mathcal{A}$ with $\pi(\hat\varphi) = 0$ and $\pi(\delta\hat\varphi) \neq 0$. By dividing out these unpleasant forms, we arrive at the desired differential algebra $\Omega_\mathcal{D}\mathcal{A}$,

$$\Omega_\mathcal{D}\mathcal{A} := \frac{\pi\left(\Omega\mathcal{A}\right)}{\mathcal{J}}, \quad \text{with} \quad \mathcal{J} := \pi\left(\delta \ker \pi\right) =: \bigoplus_p \mathcal{J}^p,$$

($\mathcal{J}$ for junk). On the quotient, the differential is now well defined. Degree by degree we have:

$$\Omega^0_\mathcal{D}\mathcal{A} = \rho(\mathcal{A})$$

because $\mathcal{J}^0 = 0$,

$$\Omega^1_\mathcal{D}\mathcal{A} = \pi(\Omega^1\mathcal{A})$$

because ρ is faithful, and in degree $p \geq 2$

$$\Omega^p_\mathcal{D}\mathcal{A} = \frac{\pi(\Omega^p\mathcal{A})}{\pi(\delta(\ker\pi)^{p-1})}.$$

In the commutative case, $\delta = \mathrm{d}$, $\Omega_{\not{\partial}}\mathcal{C}^\infty(M)$ is isomorphic to de Rham's differential algebra ΩM with

$$\pi(f_0\mathrm{d}f_1\mathrm{d}f_2...\mathrm{d}f_p) \cong (-i)^p f_0\gamma^{\mu_1}\left(\frac{\partial f_1}{\partial x^{\mu_1}}\right)\gamma^{\mu_2}\left(\frac{\partial f_2}{\partial x^{\mu_2}}\right)...\gamma^{\mu_p}\left(\frac{\partial f_p}{\partial x^{\mu_p}}\right). \tag{23}$$

Dividing out the junk renders the lhs graded commutative. The orientability axiom alluded to above is motivated from this isomorphism, $\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}x^4 \cong (\det g_{\cdot\cdot})^{1/2}\gamma^1\gamma^2\gamma^3\gamma^4 = (\det g_{\cdot\cdot})^{1/2}\gamma_5$.

3.4 The scalar products in noncommutative geometry

To play the Yang-Mills game, we need a scalar product for differential forms. In the noncommutative context, the scalar product has another utility. It allows us to interpret the differential forms in $\Omega_\mathcal{D}\mathcal{A}$ not as classes but as concrete operators on the Hilbert space $\mathcal{H}$: degree by degree, we embed $\Omega^p_\mathcal{D}$ in $\pi(\Omega^p\mathcal{A})$ as orthogonal complement of $\mathcal{J}^p$. If $\mathcal{H}$ was finite dimensional, we would naturally take as scalar product of two operators ω and κ, $<\omega,\kappa>= \mathrm{tr}\,(\omega^*\kappa)$. For infinite dimensional Hilbert spaces, we have to regularize and we use the Dirac operator to do so. Thanks to the asymptotic behavior of its spectrum, $\mathrm{tr}\,[\omega^*\kappa\,|\mathcal{D}|^{-\dim}]$ only diverges logarithmically. The Dixmier trace tr_Dix gets rid of this divergence [19] and we have a natural scalar product:

$$<\omega,\kappa>= \mathrm{Re}\ \mathrm{tr}_\mathrm{Dix}[\omega^*\kappa\,|\mathcal{D}|^{-\dim}], \quad \omega,\kappa \in \pi(\Omega^p\mathcal{A}).$$

We denote by $(\cdot,\cdot)$ its restriction to $\Omega_\mathcal{D}\mathcal{A}$. In the commutative case of a four dimensional manifold M, these scalar products are independent of M:

$$\begin{aligned} <\omega,\kappa> &= \tfrac{1}{32\pi^2}\mathrm{Re}\int_M \mathrm{tr}\,_4[\omega^*\kappa]\mathrm{d}^4x, \quad \omega,\kappa \in \pi(\Omega^p\mathcal{A}), \\ (\omega,\kappa) &= \tfrac{1}{8\pi^2}\mathrm{Re}\int_M \omega^* * \kappa, \quad \omega,\kappa \in \Omega^p_\mathcal{D}\mathcal{A}, \end{aligned}$$

where we have used the isomorphism (23) and view the quotient by the junk as subspace orthogonal to the junk. We anticipate that this scalar product will also induce the one we need on the Lie algebra. In order to get the coupling constants, we soften the scalar products to:

$$\begin{aligned} <\omega,\kappa>_z &= \mathrm{Re}\ \mathrm{tr}_{\mathrm{Dix}}[z\omega^*\kappa\,|\mathcal{D}|^{-\dim}], \quad \omega,\kappa\in\pi(\Omega^p\mathcal{A}) && (24)\\ (\omega,\kappa)_z &= \mathrm{Re}\ \mathrm{tr}_{\mathrm{Dix}}[z\omega^*\kappa\,|\mathcal{D}|^{-\dim}], \quad \omega,\kappa\in\Omega^p_{\mathcal{D}}\mathcal{A}. && (25)\end{aligned}$$

z is a positive operator on Hilbert space that commutes with ρ, $J\rho J^{-1}$, $\mathcal{D}$ and χ. Whether or not z commutes with J will be a difficult choice. In the commutative case, we have anyhow that z is proportional to the identity.

3.5 The commutative Yang-Mills action

The message of this subsection is that the commutative spectral triple of space-time M is a natural tool to reconstruct Maxwell's theory: this reconstruction unifies space-time with internal space, $G=U(1)$. The first sign for this unification comes from the group of unitaries of $\mathcal{A}$. Remember that $\mathcal{A}$ is the algebra of complex valued function on M with involution just complex conjugation. The group of unitaries $U(\mathcal{A}) := \{u\in\mathcal{A},\ uu^*=u^*u=1\}$ for this algebra is the group of functions from space-time into $U(1)$ and this is Maxwell's gauge group. Maxwell's four potential $A\in\Omega^1_{\partial\!\!\!/}\mathcal{A}$ is an anti-Hermitean 1-form on which a gauge transformation or unitary $u=\exp i\Lambda$ acts affinely by

$$A^u := \rho(u)A\rho(u^{-1}) + \rho(u)\mathrm{d}\rho(u^{-1}) = A - i\mathrm{d}\Lambda.$$

The field strength

$$F := \mathrm{d}A + A^2 = \mathrm{d}A \quad \in \Omega^2_{\partial\!\!\!/}\mathcal{A}$$

transforms homogeneouly under unitaries and is even gauge invariant in the commutative case,

$$F^u = \rho(u)F\rho(u^{-1}) = F.$$

The positive operator z from the commutant, that defines the scalar product can only be a multiple of the identity. Finally the obviously gauge invariant Maxwell's action can be written,

$$\begin{aligned} S_{\mathrm{Maxwell}}[A] &= (F,F) = \mathrm{Re}\ \mathrm{tr}_{\mathrm{Dix}}(zF^*F|\partial\!\!\!/|^{-4}) = \tfrac{1}{8\pi^2}\int_M zF^* * F\\ &= \tfrac{z}{16\pi^2}\int_M F^*_{\mu\nu}F^{\mu\nu}(\det g_{\cdot\cdot})^{1/2}\ \mathrm{d}^4x =: \tfrac{\epsilon_0}{4e^2}\int_M F^*_{\mu\nu}F^{\mu\nu}(\det g_{\cdot\cdot})^{1/2}\ \mathrm{d}^4x.\end{aligned}$$

Therefore $z=\pi\hbar c/\alpha_{\mathrm{em}}$ with the fine-structure constant $\alpha_{\mathrm{em}} := e^2/(4\pi\epsilon_0\hbar c)$. Later, we will call z noncommutative coupling constant. Had we dropped the condition that z commute with the Dirac operator, we would have inherited an x dependent coupling 'constant'.

The commutative pure Yang-Mills theory is linear and to justify the word coupling constant, we have to add matter, say an electron ψ. The Dirac operator acts on it defining its kinetic energy, unitaries act on it by

$$\psi^u = \rho(u)\psi, \quad u \in U(\mathcal{A}), \quad \psi \in \mathcal{H},$$

and we define the minimal coupling by the covariant Dirac operator $\not{D} := \not{\partial} + \pi(A)$. The Dirac action then reads

$$S_{\mathrm{Dirac}}[\psi, A] = \int_M \psi^* \not{D}\psi \, (\det g_{\cdot\cdot})^{1/2} \, \mathrm{d}^4 x,$$

where here the star denotes the dual with respect to the scalar product of the Hilbert space $\mathcal{H}$. A mass term $m_\psi \psi^* \psi$ may be added.

Let us stress again that in Connes' formulation, the gauge coupling, that is the invariant scalar product in internal space, is induced from the scalar product of differential forms over space-time.

3.6 The tensor product

One way to see the above commutative example is to say that the associative algebra of the spectral triple is $\mathcal{A}_t = \mathcal{F} \otimes \mathcal{A}_f$, a tensor product of the commutative, infinite dimensional algebra of *real* valued functions $\mathcal{C}^\infty(M)$ on space-time and the commutative, finite dimensional, *real* algebra $\mathcal{A}_f = \mathbb{C}$. The gauge group then is Abelian, $G = U(1) \subset \mathcal{A}_f$. It is natural to try noncommutative algebras for $\mathcal{A}_f$ to get non-Abelian gauge groups. In this spirit, we have to consider tensor products of entire spectral triples, and the message of this subsection is that if the fermionic representation breaks parity, the Higgs scalar and the symmetry breaking potential come free of charge.

Let us denote by $(\mathcal{F}, \mathcal{S}, \not{\partial}, \gamma_5, C)$ the commutative spectral triple of a four dimensional space-time and by $(\mathcal{A}_f, \mathcal{H}_f, \mathcal{D}_f, \chi_f, J_f)$, $\cdot_f$ for finite, the one of a (zero dimensional) internal space. Note that our C is anti-unitary. According to the rules of noncommutative geometry the tensor product of these two spectral triples $(\mathcal{A}_t, \mathcal{H}_t, \mathcal{D}_t, \chi_t, J_t)$, $\cdot_t$ for tensor, is:

$$\begin{aligned} \mathcal{A}_t &= \mathcal{F} \otimes \mathcal{A}_f, \quad & \mathcal{H}_t &= \mathcal{S} \otimes \mathcal{H}_f, \quad & \mathcal{D}_t &= \not{\partial} \otimes 1 + \gamma_5 \otimes \mathcal{D}_f, \\ & & \chi_t &= \gamma_5 \otimes \chi_f, \quad & J_t &= C \otimes J_f. \end{aligned}$$

Before turning the crank, we must talk about the internal Dirac operator $\mathcal{D}_f$. From the axioms, we infer that with respect to the decomposition (22) of the fermionic Hilbert space $\mathcal{H}_f$ the internal Dirac operator has the form:

$$\mathcal{D}_f = \begin{pmatrix} 0 & \mathcal{M} & 0 & 0 \\ \mathcal{M}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{\mathcal{M}} \\ 0 & 0 & \overline{\mathcal{M}^*} & 0 \end{pmatrix},$$

where $\mathcal{M}$ is the fermionic mass matrix. This is another manifestation of the unification of space-time and internal space, the naked Dirac operator $\partial\!\!\!/$ and its mass matrix satisfy the same list of axioms above.

As in the commutative case, we start by identifying the gauge group, the functions from space-time into the finite dimensional Lie group $G = U(\mathcal{A}_f)$. It is represented affinely on the bosonic fields. They are anti-Hermitean 1-forms. But now,

$$\Omega^1_{\mathcal{D}_t}\mathcal{A}_t = \Omega^1_{\partial\!\!\!/}\mathcal{F} \otimes \Omega^0_{\mathcal{D}_f}\mathcal{A}_f \,\oplus\, \Omega^0_{\partial\!\!\!/}\mathcal{F} \otimes \Omega^1_{\mathcal{D}_f}\mathcal{A}_f \,\cong\, \Omega^1(M,\mathcal{A}_f) \,\oplus\, \mathcal{F} \otimes \Omega^1_{\mathcal{D}_f}\mathcal{A}_f \,\ni A_t =: A \oplus H.$$

From the anti-Hermiticity of A_t, it follows that A is in fact a Lie algebra valued 1-form on space-time, $A \in \Omega^1(M,\mathfrak{g})$, i.e. a Yang-Mills potential. $\mathfrak{g} := u(\mathcal{A}_f) := \{X \in \mathcal{A}_f,\ X + X^* = 0\}$ is the Lie algebra of the group of unitaries $G = U(\mathcal{A}_f)$. On the other hand, the Higgs scalar H is a 0-form on space-time, valued in a representation of the Lie group G. The inhomogeneous transformation law,

$$\begin{aligned} A_t^u &= \rho_t(u)A_t\rho_t(u^{-1}) + \rho_t(u)\mathrm{d}_t\rho_t(u^{-1}) = A^u \oplus H^u, \\ &\ A^u = uAu^{-1} + u\mathrm{d}u^{-1}, \quad H^u = \rho_f(u)H\rho_f(u^{-1}) + \rho_f(u)\delta\rho_f(u^{-1}), \end{aligned}$$

determines according to which *group* representation the Higgs scalar transforms and this depends on the details of the internal spectral triple. We denote by ρ_t the representation of $\mathcal{A}_t$ on $\mathcal{H}_t$, by ρ_f the representation of $\mathcal{A}_f$ on $\mathcal{H}_f$, by δ_t the differential of $\Omega_{\mathcal{D}_t}\mathcal{A}_t$ and so forth. Next we define the field strength,

$$F_t = \delta_t A_t + A_t^2 \quad \in \Omega^2_{\mathcal{D}_t}\mathcal{A}.$$

To decompose the field strength, it is comfortable to change scalar variables,

$$\Phi(x) \;:=\; H(x) - i\mathcal{D}_f \;=\; -\Phi^*(x) \;\in \Omega^0(M, \Omega^1_{\mathcal{D}_f}\mathcal{A}_f).$$

This change of variables is well defined within $\Omega^0(M,\Omega^1_{\mathcal{D}_f}\mathcal{A}_f)$ thanks to the orientability axiom [7]. Φ has the good taste to transform homogeneously under a gauge transformation u and we can define its covariant exterior derivative,

$$\mathrm{D}\Phi := \mathrm{d}\Phi + [\rho_f(A), \Phi] \;\in \Omega^0(M, \Omega^1_{\mathcal{D}_f}\mathcal{A}_f).$$

The field strength decomposes as

$$F_t = F + (C - \alpha C) - \mathrm{D}\Phi\gamma_5),$$

with

$$\begin{aligned} F &= \mathrm{d}A + A^2 \;\in \Omega^2(M,\mathfrak{g}), \\ C &= \delta H + H^2 \in \Omega^0(M, \Omega^2_{\mathcal{D}_f}\mathcal{A}_f). \end{aligned}$$

The internal field strength C, for curvature, should not be confused with the C of charge conjugation. $\alpha C \in \Omega^0(M, \Omega^2_{\mathcal{D}_f}\mathcal{A}_f + \mathcal{J}_f^2)$ is the tricky piece of the computation, it comes from the interference in degree two of space-time junk and internal junk. The former is isomorphic to $\Omega^0 M$, a first happy circumstance. A second is that the positive operator z_t in the scalar product is necessarily of the form $z_t = 1 \otimes z_f$. Both circumstances together allow to compute αC pointwise [20]. For fixed x, $C \in \Omega_{\mathcal{D}_f}\mathcal{A}_f \subset \mathrm{End}\mathcal{H}_f$ and $\alpha C \in \pi(\Omega\mathcal{A}_f) \subset \mathrm{End}\mathcal{H}_f$ are finite dimensional operators, i.e. matrices.

Let us denote by $< \omega, \kappa >_{z_f} = \mathrm{Re}\,\mathrm{tr}\,[z_f \omega^* \kappa]$, the finite dimensional scalar product. Then αC is uniquely determined by the linear equations

$$< r, C - \alpha C >_{z_f} = 0 \qquad \text{for all} \quad r \in \rho_f(\mathcal{A}_f), \tag{26}$$

$$< j, C - \alpha C >_{z_f} = 0 \qquad \text{for all} \quad j \in \mathcal{J}_f^2, \tag{27}$$

where the trace is over the finite dimensional Hilbert space $\mathcal{H}_f$. Under a gauge transformation $u(x)$, the field strength transforms homogeneously and we can define, as before, the Yang-Mills action,

$$S_{\mathrm{YM}}[F_t] = (F_t, F_t)_{z_t} = \mathrm{Re}\ \mathrm{tr}_{\mathrm{Dix}}(z_t F_t^* F_t |\mathcal{D}_t|^{-4}).$$

The differential algebra contains the Lie algebra as 0-forms and the scalar product $(\cdot,\cdot)_{z_t}$ with $z_t = 1 \otimes z_f$ restricted to the Lie algebra is an invariant scalar product. Therefore this action is gauge invariant. Let us decompose it, $S_{\mathrm{YM}}[F_t] = S_{\mathrm{YM}}[F, H]$:

$$S_{\mathrm{YM}}[F,H] = \tfrac{1}{8\pi^2}\int_M < F, *F >_{z_f} + \tfrac{1}{8\pi^2}\int_M < \mathrm{D}\Phi, *\mathrm{D}\Phi >_{z_f} + \tfrac{1}{8\pi^2}\int_M *V(H),$$

with

$$V(H) = < C - \alpha C, C - \alpha C >_{z_f} = (C, C)_{z_f} - < \alpha C, \alpha C >_{z_f}\ .$$

The first term, a non-Abelian Yang-Mills action, is no surprise. The second, a Klein-Gordon action, propagates the Higgs scalar. The Higgs potential $V(H)$ breaks the gauge group spontaneously, if the fermions break parity. As we shall see, the computation of the Higgs sector, representation and potential, will be intricate even though it follows from a simple geometric definition, $S_{\mathrm{YM}}[F_t] = (F_t, F_t)_{z_t}$. This simple geometric definition constrains the ensuing Yang-Mills theory. The Lie group G is not arbitrary, it must be a group of unitaries of an associative algebra, which is not the case for the exceptional groups. Furthermore, the fermionic representation is not only a representation of the group but must also be a representation of the algebra which is not the case for representations other than the fundamental ones. Finally, the Higgs representation is computed, not chosen. In any case, no left-right symmetric and no grand unified theory admits a formulation within noncommutative geometry.

To end this subsection, we mention the Dirac Lagrangian, $\mathcal{L}_{\mathrm{Dirac}} = \psi^* \mathcal{D}_{t,\mathrm{cov}}\psi$. The total, covariant Dirac operator is

$$\mathcal{D}_{t,\mathrm{cov}} = \mathcal{D}_t + \pi_t(A_t) + J_t(\mathcal{D}_t + \pi_t(A_t))J_t^{-1}. \tag{28}$$

Note the appearance of charge conjugation that will be crucial. The decomposition of this Lagrangian is:

$$\mathcal{L}_{\rm Dirac} = \psi^*(\partial\!\!\!/ - i\rho_f(A\!\!\!/) - J_t i\rho_f(A\!\!\!/)J_t^{-1})\psi - \psi^*(\Phi\gamma_5 + J_t\Phi\gamma_5 J_t^{-1})\psi.$$

In words: noncommutative geometry promotes the Higgs scalar to a connection and thereby unifies the gauge couplings hidden in $\rho_f(A\!\!\!/)$ with the Yukawa couplings hidden in Φ.

3.7 The standard model

3.7.1 The algebraic setting

It is time for an example. Concerning its choice, we emphasize two points. The standard model has an internal space that does fit the elaborate axioms of a spectral triple. The internal spectral triple of the standard model is not far from being the simplest, non-degenerate example. To make this more precise, we note that the standard model viewed as an ordinary Yang-Mills-Higgs theory has the following four *unrelated* features:

(i) weak interactions break parity maximally,

(ii) weak interactions suffer spontaneous break down,

(iii) strong interactions do not break parity,

(iv) strong interactions do not suffer spontaneous break down.

Flip just one of these features and the standard model is outside the noncommutative axioms [21][7]. A more quantitative constraint concerns the Higgs representation, that in Connes' formulation is not chosen but computed. The spectral triple of the standard model implies that the Higgs scalar transforms as one doublet under weak isospin entailing a unit ρ-factor,

$$\rho := \frac{m_W^2}{\cos^2(\theta_w)\ m_Z^2} = 1.$$

Experimentally we have today $\rho = 1.0012 \pm 0.0031$.

The geometric version of the standard model is well documented in the literature [1][2][3][22] and we just have to fix our notations. We denote by $\mathbb{H}$ the algebra of quaternions, viewed as 2×2 matrices,

$$\begin{pmatrix} x & -\bar y \\ y & \bar x \end{pmatrix} \in \mathbb{H}, \quad x, y \in \mathbb{C}.$$

From now on, everything concerns the internal spectral triple and we drop the subscript f for finite,

$$\begin{aligned} \mathcal{A} &= \mathbb{H}\oplus\mathbb{C}\oplus M_3(\mathbb{C}) \ni (a,b,c), \\ \mathcal{H}_L &= \left(\mathbb{C}^2\otimes\mathbb{C}^N\otimes\mathbb{C}^3\right) \ \oplus\ \left(\mathbb{C}^2\otimes\mathbb{C}^N\otimes\mathbb{C}\right), \\ \mathcal{H}_R &= \left((\mathbb{C}\oplus\mathbb{C})\otimes\mathbb{C}^N\otimes\mathbb{C}^3\right) \ \oplus\ \left(\mathbb{C}\otimes\mathbb{C}^N\otimes\mathbb{C}\right). \end{aligned}$$

In each summand, the first factor denotes weak isospin doublets or singlets, the second N generations, $N = 3$, and the third denotes color triplets or singlets. Let us choose the following basis of $\mathcal{H} = \mathbb{C}^{90}$:

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L, \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L;$$

$$\begin{matrix} u_R, & c_R, & t_R, \\ d_R, & s_R, & b_R, \end{matrix} \quad e_R, \quad \mu_R, \quad \tau_R;$$

$$\begin{pmatrix} u \\ d \end{pmatrix}_L^c, \begin{pmatrix} c \\ s \end{pmatrix}_L^c, \begin{pmatrix} t \\ b \end{pmatrix}_L^c, \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L^c, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L^c, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L^c;$$

$$\begin{matrix} u_R^c, & c_R^c, & t_R^c, \\ d_R^c, & s_R^c, & b_R^c, \end{matrix} \quad e_R^c, \quad \mu_R^c, \quad \tau_R^c.$$

The representation ρ acts on $\mathcal{H}$ by

$$\rho(a,b,c) := \begin{pmatrix} \rho_w(a,b) & 0 \\ 0 & \rho_s(b,c) \end{pmatrix} := \begin{pmatrix} \rho_{wL}(a) & 0 & 0 & 0 \\ 0 & \rho_{wR}(b) & 0 & 0 \\ 0 & 0 & \overline{\rho_{sL}(b,c)} & 0 \\ 0 & 0 & 0 & \overline{\rho_{sR}(b,c)} \end{pmatrix}$$

with

$$\begin{aligned} \rho_{wL}(a) &:= \begin{pmatrix} a \otimes 1_N \otimes 1_3 & 0 \\ 0 & a \otimes 1_N \end{pmatrix}, \quad \rho_{wR}(b) := \begin{pmatrix} B \otimes 1_N \otimes 1_3 & 0 \\ 0 & \bar{b} 1_N \end{pmatrix}, \\ & B := \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix}, \\ \rho_{sL}(b,c) &:= \begin{pmatrix} 1_2 \otimes 1_N \otimes c & 0 \\ 0 & \bar{b} 1_2 \otimes 1_N \end{pmatrix}, \quad \rho_{sR}(b,c) := \begin{pmatrix} 1_2 \otimes 1_N \otimes c & 0 \\ 0 & \bar{b} 1_N \end{pmatrix}. \end{aligned}$$

The chosen representation ρ will take into account weak interactions $\rho_w(a,b)$, $a \in \mathbb{H}$, $b \in \mathbb{C}$, and strong interactions $\rho_s(b,c)$, $c \in M_3(\mathbb{C})$, c for color. This choice discriminates between leptons (color singlets) and quarks (color triplets). The role of $b \in \mathbb{C}$ appearing in both weak interactions $\rho_w(a,b)$ and strong interactions $\rho_s(b,c)$ is crucial to make $\rho(a,b,c)$ a representation of $\mathcal{A}$ and is crucial for weak hypercharge computations. There is an apparent asymmetry between particles and anti-particles, the former are subject to weak, the latter to strong interactions. However, since particles and anti-particles are permuted in the covariant Dirac operator (28) by

$$J = \begin{pmatrix} 0 & 1_{15N} \\ 1_{15N} & 0 \end{pmatrix} \circ c.c.,$$

the theory is invariant under charge conjugation. We denote the complex conjugation by *c.c.*. For completeness, we record the chirality as matrix

$$\chi = \begin{pmatrix} -1_{8N} & 0 & 0 & 0 \\ 0 & 1_{7N} & 0 & 0 \\ 0 & 0 & -1_{8N} & 0 \\ 0 & 0 & 0 & 1_{7N} \end{pmatrix}.$$

The third item in the spectral triple is the Dirac operator

$$\mathcal{D}=\begin{pmatrix}0&\mathcal{M}&0&0\\ \mathcal{M}^*&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}.$$

The fermionic mass matrix of the standard model is

$$\mathcal{M}=\begin{pmatrix}\begin{pmatrix}M_u&0\\ 0&M_d\end{pmatrix}\otimes 1_3&0\\ 0&\begin{pmatrix}0\\ M_e\end{pmatrix}\end{pmatrix},$$

with

$$M_u:=\begin{pmatrix}m_u&0&0\\ 0&m_c&0\\ 0&0&m_t\end{pmatrix},\quad M_d:=C_{KM}\begin{pmatrix}m_d&0&0\\ 0&m_s&0\\ 0&0&m_b\end{pmatrix},\quad M_e:=\begin{pmatrix}m_e&0&0\\ 0&m_\mu&0\\ 0&0&m_\tau\end{pmatrix}.$$

All indicated fermion masses are supposed positive and different. The Cabibbo-Kobayashi-Maskawa matrix C_{KM} is supposed non-degenerate in the sense that there is no simultaneous mass and weak interaction eigenstate. Note that the strong interactions are vector-like and ρ_s commutes with $\mathcal{D}$.

Let us compute the noncommutative coupling constant z. We recall that z is a positive operator on $\mathcal{H}$ that commutes with the representation ρ, with its opposite $J\rho J^{-1}$, with the chirality χ, and with the Dirac operator $\mathcal{D}$. It follows that z involves $2(1+N)=8$ strictly positive numbers $x,\ y_1,\ y_2,\ y_N,\ \tilde{x},\ \tilde{y}_1,\ \tilde{y}_2,\ \tilde{y}_N$,

$$z := \begin{pmatrix}z_w&0\\ 0&z_s\end{pmatrix},$$

$$z_w := \begin{pmatrix}x/3\,1_2\otimes 1_N\otimes 1_3&0&0&0\\ 0&1_2\otimes y&0&0\\ 0&0&x/3\,1_2\otimes 1_N\otimes 1_3&0\\ 0&0&0&y\end{pmatrix},$$

$$z_s := \begin{pmatrix}\tilde{x}/3\,1_2\otimes 1_N\otimes 1_3&0&0&0\\ 0&1_2\otimes \tilde{y}&0&0\\ 0&0&\tilde{x}/3\,1_2\otimes 1_N\otimes 1_3&0\\ 0&0&0&\tilde{y}\end{pmatrix},$$

$$y := \begin{pmatrix}y_1&0&0\\ 0&y_2&0\\ 0&0&y_N\end{pmatrix},\quad \tilde{y}:=\begin{pmatrix}\tilde{y}_1&0&0\\ 0&\tilde{y}_2&0\\ 0&0&\tilde{y}_N\end{pmatrix}.$$

The interpretation of these numbers is straightforward. The three y_j poise the weak interactions with the three lepton generations. The y_j enter independently because the Higgs scalar couples differently to the three leptons and in noncommutative geometry the Higgs is part of the gauge

interactions. The three $\tilde{y}_j$ poise the 'strong' interactions with the three lepton generations. They do not drop out because of the b in ρ_s. However, they will only enter as sum: strong interactions are unbroken and do not generate a Higgs. We will denote $\tilde{y} := \tilde{y}_1 + \tilde{y}_2 + \tilde{y}_N$ and there should be no risk of confusion. x and $\tilde{x}$ poise weak and strong interactions with quarks. There is only one number per interaction because of the Cabibbo-Kobayashi-Maskawa mixing that we suppose non-degenerate.

3.7.2 The choice of a scalar product

We recall the internal scalar product $<\omega,\kappa>_z = \mathrm{Re}\,\mathrm{tr}\,[z\omega^*\kappa]$, $\omega,\ \kappa \in \pi(\Omega\mathcal{A})$. At this point comes the new lesson from noncommutative relativity. It tells us that we have forgotten an entire cone of other scalar products,

$$<\omega,\kappa>_{z'} := \ \mathrm{Re}\,\mathrm{tr}\,[z'(\omega + J\omega J^{-1})^*(\kappa + J\kappa J^{-1})]$$

with additional $1+N$ strictly positive constants $x',\ y_1',\ y_2',\ y_N'$,

$$\begin{aligned} z' &:= \begin{pmatrix} z_w' & 0 \\ 0 & z_s' \end{pmatrix}, \\ z_w' &= z_s' := \tfrac{1}{2}\begin{pmatrix} x'/3\, 1_2\otimes 1_N\otimes 1_3 & 0 & 0 & 0 \\ 0 & 1_2\otimes y' & 0 & 0 \\ 0 & 0 & x'/3\, 1_2\otimes 1_N\otimes 1_3 & 0 \\ 0 & 0 & 0 & y' \end{pmatrix}. \end{aligned}$$

Indeed, in noncommutative relativity, the scalar product is not chosen, it is induced from the heat kernel calculation. The Dirac operator $\mathcal{D}_{t,\mathrm{cov}} = \mathcal{D}_t + \pi_t(A_t) + J_t\pi_t(A_t)J_t^{-1}$ leads to the scalar product with z'. We could obtain the one with z from another Dirac operator, $\mathcal{D}_{t,\mathrm{cov}} = \mathcal{D}_t + \pi_t(A_t)$, but this latter is forbidden by the spectral principle: for a unitary $u \in \mathcal{A}_t$, the inner automorphim

$$\alpha_u : \rho_t(a) \in \rho_t(\mathcal{A}_t) \longrightarrow \rho_t(uau^*) \in \rho_t(\mathcal{A}_t)$$

induces a unitary operator $U = uJuJ^{-1}$ on $\mathcal{H}_t$ satisfying

$$U\rho_t(a)U^* = \alpha_u(\rho_t(a)),\ \text{and}\ U\mathcal{D}_tU^* = \mathcal{D}_t + A + JAJ^{-1}\ \text{with}\ A = u[\mathcal{D}_t, u^*],$$

so $\mathcal{D}_t$ and $\mathcal{D}_{t,cov}$ have the same spectrum and the fluctuations of the metric are of the form $A + JAJ^{-1}$.

Restricted to the Lie algebra $\mathfrak{g}$, we have a subtle nuance between the two invariant scalar products concerning the two $u(1)$ factors. For z' in the center $\mathbb{R}^+1$, we have $\sin^2\theta_w = 3/8$, while for z in the center we will get $\sin^2\theta_w = 12/29$. Note also that z' commutes with the real structure J, while z does not. If in doubt, stay out: we will use both cones simultaneously. Figure 2 is an artist's view on the role of the possible scalar products in noncommutative Yang-Mills theory:

$$<\omega,\kappa>_{z,z'} := \mathrm{Re}\,\mathrm{tr}\,[z\omega^*\kappa] + \mathrm{Re}\,\mathrm{tr}\,[z'(\omega + J\omega J^{-1})^*(\kappa + J\kappa J^{-1})], \quad \omega,\ \kappa \in \pi(\Omega\mathcal{A}). \qquad (29)$$

3.7.3 The gauge couplings computation

We are ready to turn the crank. A long, but straight down the line computation leads to the physical couplings in terms of the fermionic mass matrix $\mathcal{M}$ and the noncommutative couplings $z,\ z'$:

Here are a few purely algebraic intermediate steps:

$$\Omega^1_{\mathcal{D}}\mathcal{A} = \left\{ i \begin{pmatrix} 0 & \rho_{wL}(h)\mathcal{M} & 0 & 0 \\ \mathcal{M}^*\rho_{wL}(\tilde{h}^*) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ h, \tilde{h} \in \mathbb{H} \right\}.$$

The Higgs being an anti-Hermitian 1-form

$$H = i \begin{pmatrix} 0 & \rho_{wL}(h)\mathcal{M} & 0 & 0 \\ \mathcal{M}^*\rho_L(h^*) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} h_1 & -\bar{h}_2 \\ h_2 & \bar{h}_1 \end{pmatrix} \in \mathbb{H}$$

is parameterized by one complex doublet

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad h_1, h_2 \in \mathbb{C}.$$

The internal junk in degree two turns out to be

$$\mathcal{J}^2 = \left\{ i \begin{pmatrix} j \otimes \Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad j \in \mathbb{H} \right\}$$

with

$$\Delta := \tfrac{1}{2} \begin{pmatrix} (M_u M_u^* - M_d M_d^*) \otimes 1_3 & 0 \\ 0 & -M_e M_e^* \end{pmatrix}.$$

The homogeneous scalar variable is:

$$\Phi := H - i\mathcal{D} =: i \begin{pmatrix} 0 & \rho_{wL}(\phi)\mathcal{M} & 0 & 0 \\ \mathcal{M}^*\rho_{wL}(\phi^*) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \phi = \begin{pmatrix} \varphi_1 & -\bar{\varphi}_2 \\ \varphi_2 & \bar{\varphi}_1 \end{pmatrix} \in \mathbb{H},$$

and with $\varphi := (\varphi_1, \varphi_2)^T$, the internal field strength is:

$$C \ := \ \delta H + H^2 = \left(1 - |\varphi|^2\right) \begin{pmatrix} 1_2 \otimes \Sigma & 0 & 0 & 0 \\ 0 & \mathcal{M}^*\mathcal{M} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Sigma := \tfrac{1}{2} \begin{pmatrix} (M_u M_u^* + M_d M_d^*) \otimes 1_3 & 0 \\ 0 & M_e M_e^* \end{pmatrix}.$$

Now, the chosen scalar product (29) appears in the long computation of αC from (26) and (27). In the standard model, equation (27) implies that αC has no junk component, and has the form

$$\alpha C = (1-|\varphi|^2)m_t^2\,\rho(\alpha 1_2,\beta,\gamma 1_3).$$

To compute the real numbers $\alpha,\ \beta,\ \gamma$, we neglect all fermion masses with respect to the top mass. This approximation is again good to $m_b^2/m_t^2 = 0.0006$ [23]. In this approximation, the number of parameters reduces again, we are left with six parameters: $x,\ \tilde{x},\ y := y_1+y_2+y_N,\ x'$, and $y' := y_1'+y_2'+y_N', \tilde{y}$, and $\alpha,\ \beta,\ \gamma$ are determined by three linear equations:

$$\begin{array}{rcrcrcr} [N(x+x')+y+y']\alpha & + & y'\,\beta & + & Nx'\,\gamma & = & \frac{1}{2}(x+x') \\ 2y'\,\alpha & + & [2N(x+x')+y+3\tilde{y}+6y']\beta & + & 2Nx'\,\gamma & = & (x+x') \\ Nx'\,\alpha & + & Nx'\,\beta & + & 2N(\tilde{x}+x')\gamma & = & x'. \end{array} \tag{30}$$

The Higgs Lagrangian has the form:

$$\begin{aligned} &\tfrac{2}{8\pi^2}(x+x')m_t^2\,(\mathrm{D}_\mu\varphi)^*\mathrm{D}^\mu\varphi \;+\; \tfrac{1}{8\pi^2}k\,m_t^4\left(1-|\varphi|^2\right)^2 \\ &\quad =: \tfrac{1}{2}\,(\mathrm{D}_\mu\varphi_{\mathrm{ph}})^*\mathrm{D}^\mu\varphi_{\mathrm{ph}} \;+\; \lambda|\varphi_{\mathrm{ph}}|^4 \;-\; \tfrac{1}{2}\,\mu^2|\varphi_{\mathrm{ph}}|^2 + \text{constant}, \end{aligned}$$

with

$$\begin{aligned} k \;:=\;& \tfrac{3}{2}(x+x') - 2Nx(\alpha^2+\beta^2) - y(2\alpha^2+\beta^2) - 4N\tilde{x}\gamma^2 \\ & -3\tilde{y}\beta^2 - 2Nx'((\alpha+\gamma)^2+(\beta+\gamma)^2) - y'(2(\alpha+\beta)^2+4\beta^2). \end{aligned}$$

Therefore

$$\lambda^{-1} \;=\; \tfrac{2}{\pi^2}\frac{(x+x')^2}{k}, \tag{31}$$

$$\mu^2 \;=\; \frac{k}{x+x'}\,m_t^2. \tag{32}$$

Before computing the gauge couplings, we have to get rid of the unwanted $u(1)$ in $\mathfrak{g}$. This is done by imposing the unimodularity condition,

$$\mathrm{tr}\,\left[P\left(\rho(a,b,c)+J\rho(a,b,c)J^{-1}\right)\right] = 0,$$

where P is the projection on $\mathcal{H}_L \oplus \mathcal{H}_R$, the space of particles. Note that this condition is equivalent to the condition of vanishing gauge anomalies [22]. Normalizing properly the gauge fields, we compute their couplings:

$$g_3^{-2} \;=\; \tfrac{1}{6\pi^2}N(\tilde{x}+x'), \tag{33}$$

$$g_2^{-2} \;=\; \tfrac{1}{8\pi^2}[N(x+x')+y+y'], \tag{34}$$

$$g_1^{-2} \;=\; \tfrac{1}{8\pi^2}[Nx+\tfrac{2}{9}N\tilde{x}+\tfrac{11}{9}Nx'+\tfrac{1}{2}y+\tfrac{3}{2}\tilde{y}+3y']. \tag{35}$$

3.8 Results

As for noncommutative relativity, we interpret the five constraints (31-35) in terms of running quantities at the noncommutative scale Λ. Since the flow of μ^2 is renormalization scheme dependent, we trade the running top mass for its Yukawa coupling, $m_t = g_t v$, $m_W = \frac{1}{2} g_2 v$, $m_H = 2\sqrt{2\lambda}\, v$, $v = \frac{1}{2}\frac{\mu}{\sqrt{\lambda}}$,

$$
\begin{aligned}
g_3(\Lambda)^{-2} &= \tfrac{1}{6\pi^2} N(\tilde{x} + x'), && (36)\\
g_2(\Lambda)^{-2} &= \tfrac{1}{8\pi^2} [N(x + x') + y + y'], && (37)\\
g_1(\Lambda)^{-2} &= \tfrac{1}{8\pi^2} [Nx + \tfrac{2}{9} N\tilde{x} + \tfrac{11}{9} Nx' + \tfrac{1}{2} y + \tfrac{3}{2}\tilde{y} + 3y'], && (38)\\
\lambda(\Lambda)^{-1} &= \tfrac{2}{\pi^2} \frac{(x + x')^2}{k}, && (39)\\
g_t(\Lambda)^{-2} &= \tfrac{1}{2\pi^2} (x + x'). && (40)
\end{aligned}
$$

These Yang-Mills constraints are to be compared to the soft Einstein-Hilbert constraints

$$
\begin{aligned}
g_3(\Lambda)^{-2} &= \tfrac{1}{9\pi^2} f_4 N x', && (41)\\
g_2(\Lambda)^{-2} &= \tfrac{1}{12\pi^2} f_4 (N x' + y'), && (42)\\
g_1(\Lambda)^{-2} &= \tfrac{1}{12\pi^2} f_4 (\tfrac{11}{9} N x' + 3y'), && (43)\\
\lambda(\Lambda)^{-1} &= \tfrac{1}{\pi^2} f_4\, x'. && (44)
\end{aligned}
$$

The noncommutative Yang-Mills action has four additional parameters, x, $\tilde{x}$, y, $\tilde{y}$, but one additional constraint, on the top mass.

These results can be detailed at different levels, playing with z and z'.

• The original Connes-Lott model [2][1] used $z_w =: z_1$, $z_s =: z_2$, $z' = 0$ and $\Lambda = m_Z$, i.e. tree level. It worked with a bimodule and had *two* spurious $U(1)$ factors. Consequently its linear system (30) is slightly different, put $\tilde{y} = 0$, and the Higgs mass comes out [24] :

$$
\begin{aligned}
m_H^2 &= 3\, \frac{(m_t/m_W)^4 + 2(m_t/m_W)^2 - 1}{(m_t/m_W)^2 + 3}\; m_W^2,\\
m_H &= 278 \text{ GeV},
\end{aligned}
$$

for $m_t = 175$ GeV.

• With the real structure [3], we are inflicted with only one spurious $U(1)$. If we put $z' = 0$, then we can solve the system (30) even without the approximation of a dominating top mass:

$$
\alpha = \tfrac{1}{2} \frac{x}{Nx + y}, \quad \beta = \tfrac{1}{2} \frac{x}{Nx + \frac{1}{2} y + \frac{3}{2}\tilde{y}}, \quad \gamma = 0,
$$

and at tree level [23]:

$$
\begin{aligned}
m_H^2 &= 3\, {m_t}^2 - \left(1 + \frac{{g_2}^{-2}}{{g_1}^{-2} - \frac{1}{6}{g_3}^{-2}}\right) m_W^2,\\
m_H &= 289 \text{ GeV}
\end{aligned}
$$

for $m_t = 175$ GeV. In this case, we also have precise results with all fermion masses and mixings. The τ mass renders the Higgs mass fuzzy with a relative uncertainty of the order of m_τ^2/m_t^2, that is some tens of MeV, as y_3 ranges from 0 to its maximal value.

• Our general analysis including z, z' and Λ starts with the inequality,

$$g_2(\Lambda) < \tfrac{2}{\sqrt{N}} g_t(\Lambda) \tag{45}$$

coming from equations (37) and (40). Identifying the pole masses of the W and top with their running masses at m_Z, *this inequality sets an upper bound* Λ_{max} on the noncommutative scale shown in Figure 3. This bound is rather sensitive to variations in the gauge couplings. The noncommutative Einstein-Hilbert action needs a scale Λ of at least 10^{10} GeV forcing a high top mass or slightly different gauge couplings as suggested anyhow by the stiff action.

In the presence of z, the top mass is a free parameter and z' is just a perturbation rendering the Higgs mass fuzzy. This comes from the fact that x' and y' are bounded from above,

$$\begin{aligned} x'_{\text{max}} &= \min\{2\pi^2 g_3(\Lambda)^{-2},\, 2\pi^2 g_t(\Lambda)^{-2}\} \\ y'_{\text{max}} &= \min\{8\pi^2 g_2(\Lambda)^{-2} - 6\pi^2 g_t(\Lambda)^{-2},\, \tfrac{16\pi^2}{5} g_1(\Lambda)^{-2} - \tfrac{8\pi^2}{5} g_2(\Lambda)^{-2} - \tfrac{8\pi^2}{15} g_3(\Lambda)^{-2} - \tfrac{6\pi^2}{5} g_t(\Lambda)^{-2}\} \end{aligned}$$

and that the Higgs mass decreases with x', increases with y'. With $m_t = 175 \pm 6$ GeV, this fuzziness is:

$$\begin{aligned} m_H &= 289^{+2}_{-5} \text{ GeV} \quad \text{for} \quad \Lambda = m_Z, \\ m_H &= 195^{+.0}_{-.5} \text{ GeV} \quad \text{for} \quad \Lambda = \Lambda_{\text{max}} = 2 \cdot 10^5 \text{ GeV}. \end{aligned} \tag{46}$$

This narrow interval accessible to the Higgs mass comes of course from a narrow interval accessible to the scalar selfcoupling,

$$\begin{aligned} \lambda(\Lambda) &= (0.329 \,-\, 0.345)\, g_3^2 \quad \text{for} \quad \Lambda = m_Z, \\ \lambda(\Lambda) &= (0.313 \,-\, 0.317)\, g_3^2 \quad \text{for} \quad \Lambda = \Lambda_{\text{max}} = 2 \cdot 10^5 \text{ GeV}. \end{aligned}$$

Finally, we have one other constraint in presence of z and z',

$$\tfrac{1}{15} g_3(\Lambda)^{-2} + \tfrac{1}{5} g_2(\Lambda)^{-2} + \tfrac{3}{20} g_t(\Lambda)^{-2} \;<\; \tfrac{2}{5} g_1(\Lambda)^{-2}.$$

We already had this same inequality [23] with $z' = 0$. For $\Lambda = m_Z$ it means $\sin^2\theta_w < 0.54$ and remains harmless for higher Λ.

• To make contact with the noncommutative Einstein-Hilbert action, we put $z = 0$. Now the constraints on the gauge coupling are identical to those from the Einstein-Hilbert action and force upon us the big desert. In addition, the top mass is constrained,

$$g_t^2 = \tfrac{N}{3} g_3^2.$$

To compute the Higgs mass, we solve the system (30), which is simple due to the approximation of a dominating top mass:

$$\alpha = 0, \quad \beta = 0, \quad \gamma = \frac{1}{2N}\,.$$

From equation (39) we have

$$\lambda(\Lambda) = \tfrac{3N-2}{24}\, g_3^2 = \tfrac{7}{24}\, g_3^2. \tag{47}$$

This constraint is to be compared to the one from the Einstein-Hilbert action (44), $\lambda(\Lambda) = \frac{N}{9} g_3^2$. The two scalar selfcouplings would coincide precisely if we had $N = 6$ generations!

In terms of masses, we get in the soft case, $\Lambda = 0.96 \cdot 10^{10}$ GeV:

$$m_t = 214 \pm 0 \pm 4 \text{ GeV}, \tag{48}$$

$$m_H = 227 \pm 0 \pm 4 \text{ GeV} \qquad \text{soft YM}, \tag{49}$$

and in the stiff case, $z' \in \rho(\text{center})$:

$$m_t = 188 \pm 14 \pm 2 \pm 0 \text{ GeV}, \tag{50}$$

$$m_H = 198 \pm 8 \pm 2 \pm 0 \text{ GeV} \qquad \text{stiff YM}. \tag{51}$$

For comparison, we recall the values from the Einstein-Hilbert action:

$$m_H = 190 \pm 0 \pm 1 \pm 4 \text{ GeV} \qquad \text{soft EH}, \tag{52}$$

$$m_H = 182 \pm 10 \pm 2 \pm 7 \text{ GeV} \qquad \text{stiff EH}. \tag{53}$$

The first error is from the uncertainty in the noncommutative scale, $\Lambda = (10^{13} - 10^{17})$ GeV, the second from the present experimental uncertainty in the gauge couplings, $g_3 = 1.218 \pm 0.026$, and the third is from the uncertainty in the top mass, if needed as input, $m_t = 175 \pm 6$ GeV.

We give up the soft noncommutative Yang-Mills model since its top mass (48) must be bigger than 197 GeV, Figure 3. It is too large and we retain only the stiff model. The stiff values for the top mass from Yang-Mills (50) seem in contradiction with the asymptotic value 197 GeV from Figure 3. They are not, the asymptotic value is sensitive to changes in the gauge couplings which in the stiff case deviate slightly from the experimental values. The error of ± 14 GeV from the uncertainty in Λ tells us how close we are to the rim. Concerning the Higgs mass, the two allowed intervals, (53) from relativity and (51) from Yang-Mills have a non-empty intersection,

$$m_H = 188 - 201 \text{ GeV}.$$

This is the second pillar of the bridge we propose. All values of top and Higgs masses (48-53) are compatible with perturbation and stability in the energy range, $m_Z < E < \Lambda$, Figure 6.

4 Conclusions

There is an old hand waving argument combining Heisenberg's uncertainty relation with the Schwarzschild radius which implies that the noncommutative scale Λ must be smaller than the

Planck mass, 10^{19} GeV. This is compatible with the numbers above and we have the following picture. Grand unification proposed the big desert, from the Higgs mass all the way up to Λ nothing new happens, no new particle, not even a break down of perturbation theory [25]. At Λ, a modest change of physics happens. The standard $SU(3)\times SU(2)\times U(1)$ is unified into $SU(5)$. This adds a few more Yang-Mills and Higgs bosons to our boring world. These ‘lepto-quarks’ cause proton decay and make our world too exciting to be stable. Noncommutative geometry also has the big desert, but on its other side a revolution in form of a truly noncommutative space-time of which the cheap tensor product between differential geometry and the internal space of the standard model is only a low energy mirage. We expect that crossing the scale Λ will induce ‘noncommutative’ threshold effects that are responsible for the small mismatch in the gauge couplings triangle in the E-g plane (Figure 1) and for the small mismatch in the scalar selfcoupling and in the Yukawa coupling of the top. We also hope that the new geometry beyond Λ will solve our conceptual problems with quantum field theory, in particular in presence of gravity. But this is still troubled water.

The dream to connect general relativity and Yang-Mills theories is as old as Einstein. Elegant attempts have been proposed, Kaluza-Klein theories, Weyl’s gravity, Poincaré gravity, Sakharov’s induced gravity... Noncommutative geometry proposes another bridge, that stands so far thanks to a subtle conspiration between the gauge couplings, the top mass and the number of generations. But it will fall soon if the Higgs mass does not cooperate.

5 Appendix

• **Gauge couplings:** The $SU(2)\times U(1)\times SU(3)$ gauge couplings g_3, g_1, g_2 are normalized by the scalar product

$$< \rho(a,b,c), \rho(a,b,c) >_{z,z'} \;=\; \frac{1}{2} g_1^{-2} b\bar{b} + g_2^{-2} \mathrm{tr}\,(a^* a) + g_3^{-2} \mathrm{tr}\,(c^* c),$$

for $(a,b,c) \in su(2) \oplus u(1) \oplus su(3)$.

• **Higgs field:** The kinetic term of the scalar field ϕ is normalized to $\frac{1}{2}$ in the Lagrangian which is written as

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^* \partial_\mu \phi + \lambda (\phi^* \phi)^2 - \frac{\mu^2}{2} (\phi^* \phi) + ...$$

Moreover, if v is its expectation value, the relations between the gauge couplings $g_\cdot$, the selfcoupling λ and the W, top, Higgs running masses are defined by

$$\begin{aligned} v &= 2 g_2^{-1}\, m_W, \\ \lambda &= \frac{g_2^2}{32} \frac{m_H^2}{m_W^2}, \\ \mu &= \frac{1}{\sqrt{2}}\, m_H, \\ g_t &= v^{-1}\, m_t. \end{aligned}$$

All these relations depend on the energy.

• **Renormalization procedure:** We adopt the mass independent $\overline{MS}$ renormalization scheme in the approximation where all fermions masses are neglected but the top quark mass m_t. As running parameter associated to the energy E, we choose $r = \log_{10}(\frac{E}{m_Z})$. For the renormalization flow, the one-loop evolution equations of the above variables are the following first order differential equations

$$C\ g_1'(r) = \frac{41}{6} g_1(r)^3, \tag{54}$$

$$C\ g_2'(r) = -\frac{19}{6} g_2(r)^3, \tag{55}$$

$$C\ g_3'(r) = -7 g_3(r)^3, \tag{56}$$

$$C\ g_t'(r) = g_t(r)(-\frac{17}{12} g_1(r)^2 - \frac{9}{4} g_2(r)^2 - 8 g_3(r)^2 + 9 g_t(r)^2) \tag{57}$$

$$\begin{aligned} C\ \lambda'(r) &= \lambda(r)(-3g_1(r)^2 - 9g_2(r)^2 + 24g_t(r)^2 + 96\lambda(r)) \\ &\quad + \frac{3}{32} g_1(r)^4 + \frac{9}{32} g_2(r)^4 - 6g_t(r)^4 + \frac{3}{16} g_1(r)^2 g_2(r)^2, \end{aligned} \tag{58}$$

$$C\ \mu'(r) = \mu(r)(-\frac{3}{4} g_1(r)^2 - \frac{9}{4} g_2(r)^2 + 6g_t(r)^2 + 24\lambda(r)), \tag{59}$$

with $C = \frac{16\pi^2}{\ln(10)}$.

• **Initial conditions:** At $r = 0$, that is, at $m_Z = 91.187$ GeV, we have [27]

$$g_1(0) = 0.3575 \pm 0.0001, \tag{60}$$

$$g_2(0) = 0.6507 \pm 0.0007, \tag{61}$$

$$g_3(0) = 1.218 \pm 0.0026, \tag{62}$$

$$m_t(0) = 175 \pm 6 \text{ GeV}, \tag{63}$$

$$m_W(0) = 80.33 \pm 0.15 \text{ GeV}. \tag{64}$$

So we get for the central values

$$v(0) = 246.903 \text{ GeV},$$

$$g_t(0) = 0.0040.$$

The last equation (59) decouples from the others: note that $g_1, g_2, g_3, g_t, \lambda$ have no dimension while μ is a mass. At this point, it is important to quote that in our renormalization scheme, quadratic divergences do not appear, only the logarithmic ones are retained. In order to avoid renormalization scheme ambiguities in the evolution of μ^2, we neglect the threshold effects of the top and Higgs masses and we identify their pole masses $m_p = m(m_p)$ with their running masses at the Z mass $m(m_Z)$. For these reasons, we will not use (59).

• **Figures:** All quantities not explicitely mentioned in a figure are put to their experimental values.

In Figure 1, the intersections of the three coupling constants determine a triangle. Figure 4 shows the evolution of g_t. Clearly, the same evolution for the Higgs selfcoupling λ strongly

depends in Figure 5 of the Higgs initial value mass. This is due to the fact that the allowed domain for the Higgs mass in term of the top mass (Figure 6) is the slice between the top curve which describes the perturbative (or triviality) condition ($\lambda < 1$) and the down curve which describes the instability condition ($\lambda > 0$) for energies between the Z mass and the Planck mass. Naturally, this slice depends on the choice of Λ. The three points in Figure 6 are the initial conditions of the curves of Figure 5. The upper point corresponds to the upper curve which is non perturbative and the lower point corresponds to the lower curve which is unstable. Since the experimental top mass (63) localizes the Higgs mass in a very narrow region, this figure is particularly significant if we believe in perturbation and stability throughout the big desert. It is important to note here that these two assumptions traditionally imposed by hand in a classical Yang-Mills-Higgs theory, are automatically satisfied in both noncommutative dreisätze: once the initial conditions (60-64) are admitted, the selfcoupling λ always stays in the stable and perturbative regime for energies between m_Z and Λ. Note for instance that, in the soft case, $\Lambda = 0.96 \cdot 10^{10}$ GeV is small compared to the Planck mass. Figure 3 shows again that the noncommutative cutoff $\Lambda_{\rm max}$ is very sensitive to a top mass around 197 GeV.

References

[1] A. Connes, *Noncommutative Geometry*, Academic Press (1994)

[2] A. Connes & J. Lott, *The metric aspect of noncommutative geometry*, in the proceedings of the 1991 Cargèse Summer Conference, eds.: J. Fröhlich et al., Plenum Press (1992)

[3] A. Connes, *Noncommutative geometry and reality*, J. Math. Phys. 36 (1995) 6194

[4] A. Connes, *Gravity coupled with matter and the foundation of noncommutative geometry*, hep-th/9603053, Comm. Math. Phys. 155 (1996) 109

[5] A. Chamseddine & A. Connes, *The spectral action principle*, hep-th/9606001

[6] B. Iochum & T. Schücker, *Yang-Mills-Higgs versus Connes-Lott*, hep-th/9501142, Comm. Math. Phys. 178 (1996) 1

[7] T. Krajewski, *Classification of finite spectral triples*, hep-th/9701081

[8] M. Paschke & A. Sitarz, *Discrete spectral triples and their symmetries*, q-alg/9612029

[9] for other approaches see:
M. Dubois-Violette, R. Kerner & J. Madore, *Gauge bosons in a noncommutative geometry*, Phys. Lett. 217B (1989) 485
J. Madore, *An Introduction to Noncommutative Differential Geometry and its Physical Applications*, Cambridge University Press (1995)

R. Coquereaux, G. Esposito-Farèse & G. Vaillant, *Higgs fields as Yang-Mills fields and discrete symmetries*, Nucl. Phys. B353 (1991) 689

[10] A. Connes, *Noncommutative geometry and physics*, in the proceedings of the 1992 Les Houches Summer School, eds.: B. Julia, J. Zinn-Justin, North Holland (1995)
D. Kastler, *The Dirac operator and gravitation*, Comm. Math. Phys. 166 (1995) 633
W. Kalau & M. Walze, *Gravity, non-commutative Geometry and the Wodzicki residue*, gr-qc/9312031, J. Geom. Phys. 16 (1995) 327
for other approaches see:
J. Madore, *Kaluza-Klein aspects of noncommutative geometry*, in the proceedings of the 1988 conference on *Differential Geometric Methods in Theoretical Physics*, ed.: A. Solomon, World Scientific (1989)
A. Chamseddine, J. Fröhlich & O. Grandjean, *The gravitational sector in the Connes-Lott formulation of the standard model*, hep-th/9503093, J. Math. Phys. 36 (1995) 6255
T. Ackermann & J. Tolksdorf, *Unification of gravity and Yang-Mills-Higgs gauge theories*, hep-th/9503180
T. Ackermann, *Dirac operators and Clifford geometries - new unifying principles in particle physics?*, hep-th/9605129
J. Tolksdorf, *The Einstein-Hilbert-Yang-Mills-Higgs action and the Dirac-Yukawa operator*, hep-th/9612149
H. Figueroa, J. M. Gracia-Bondía, F. Lizzi & J. C. Várilly, *A nonperturbative form of the spectral action principle in noncommutative geometry*, hep-th/9701179

[11] P. B. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem*, Publish or Perish (1984)
S. A. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time*, Cambridge University Press (1989)

[12] J. Madore, *Geometric methods in classical field theory*, Phys. Rep. 75 (1981) 125
G. Vilkovisky, *The gospel according to DeWitt*, in *Quantum Theory of Gravity*, ed.: S. M. Christensen, Adam Hilger (1984)

[13] A. D. Sakharov, *Spectral density of eigenvalues of the wave equation and vacuum polarization*, Teor. Mat. Fiz. 23 (1975) 178

[14] A. D. Sakharov, *Collected Works*, Marcel Dekker (1982)

[15] U. Bonse & T. Wroblewski, *Measurement of neutron quantum interference in noninertial frames*, Phys. Rev. Lett. 1 (1983) 1401

[16] A. Connes & H. Moscovici, *The local index formula in noncommutative geometry*, Geometric And Functional Analysis 5 (1995) 174

[17] R. Estrada, J. M. Gracia-Bondía & J. C. Várilly, *On summability of distributions and spectral geometry*, funct-an/9702001

[18] L. Carminati, B. Iochum, D. Kastler & T. Schücker, *On Connes' new principle of general relativity: can spinors hear the forces of space-time?*, hep-th/9612228

[19] A. Connes, *The action functional in non-commutative geometry*, Comm. Math. Phys. 117 (1988) 673

[20] T. Schücker & J.-M. Zylinski, *Connes' model building kit*, hep-th/9312186, J. Geom. Phys. 16 (1994) 1

[21] R. Asquith, *Non-commutative geometry and the strong force*, hep-th/9509163, Phys. Lett. B 366 (1996) 220

[22] D. Kastler, *A detailed account of Alain Connes' version of the standard model in non-commutative geometry, I and II*, Rev. Math. Phys. 5 (1993) 477
D. Kastler, *A detailed account of Alain Connes' version of the standard model in non-commutative geometry, III*, Rev. Math. Phys. 8 (1996) 103
D. Kastler & M. Mebkhout, *Lectures on Non-Commutative Differential Geometry*, World Scientific, to be published
J. C. Várilly & J. M. Gracia-Bondía, *Connes' noncommutative differential geometry and the standard model*, J. Geom. Phys. 12 (1993) 223
C. P. Martín, J. M. Gracia-Bondía & J. C. Várilly, *The standard model as a non-commutative geometry: the low mass regime*, hep-th/9605001, Phys. Rep., to appear

[23] L. Carminati, B. Iochum & T. Schücker, *The noncommutative constraints on the standard model à la Connes*, hep-th/9604169, J. Math. Phys. 38 (1997) 1269

[24] D. Kastler & T. Schücker, *The standard model à la Connes-Lott*, hep-th/9412185, J. Geom. Phys. 388 (1996) 1
D. Kastler & T. Schücker, *A detailed account of Alain Connes' version of the standard model in non-commutative geometry, IV*, Rev. Math. Phys., 8 (1996) 205

[25] N. Cabibbo, L. Maiani, G. Parisi & R. Petronzio, *Bounds on the fermions and Higgs boson masses in grand unified theories*, Nucl. Phys. B158 (1979) 295

[26] C. Ford, D.R.T. Jones, P.W. Stephenson & M.B. Einhorn, *The effective potential and the renormalisation group*, Nucl. Phys. B 395 (1993) 17
B. Schrempp & M. Wimmer, *Top quark and Higgs Boson Masses: Interplay between Infrared and Ultraviolet Physics*, Progress in Particle and Nuclear Physics 37 (1996)

[27] Particule Data Group, *Review of Particles Physics*, Phys. Review D 54 (1996) 1

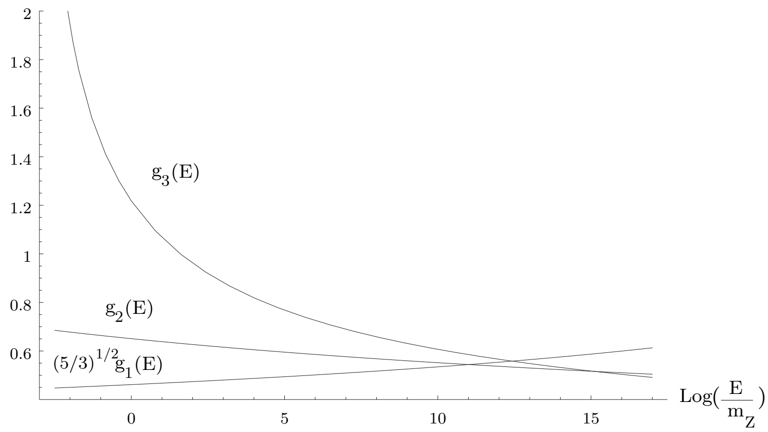

Figure 1: The evolution of the three coupling constants

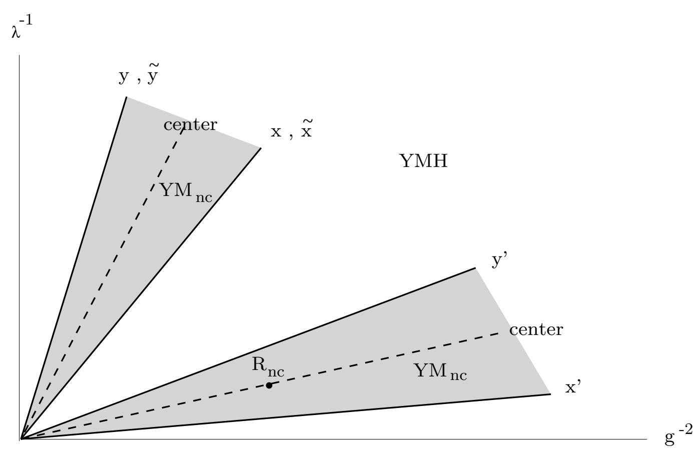

Figure 2: The allowed cones of possible scalar products

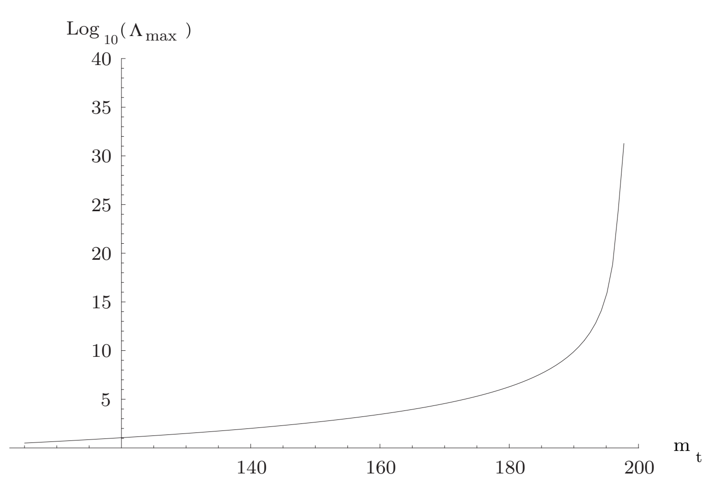

Figure 3: The cutoff $\Lambda_{\max}$ as function of the top mass

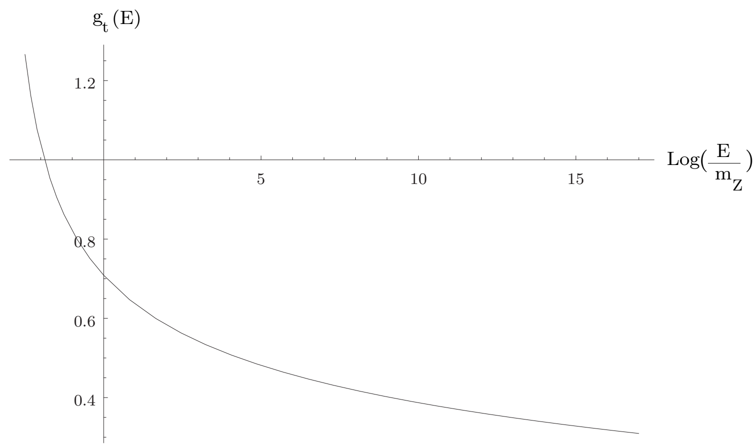

Figure 4: The top coupling

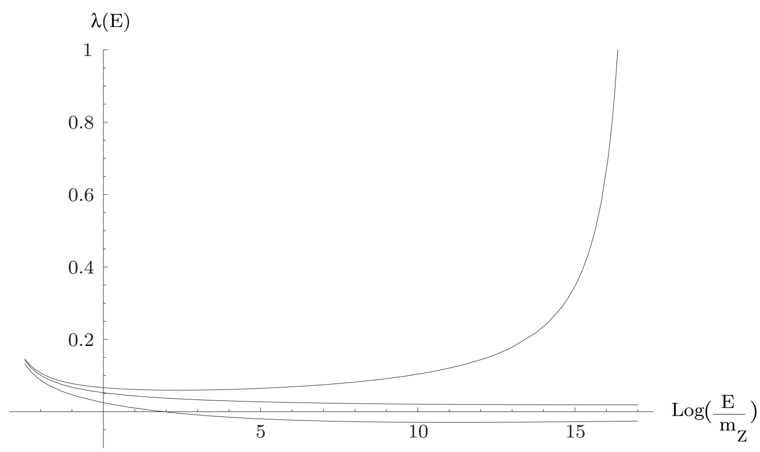

Figure 5: The Higgs selfcoupling for $m_H(m_Z) = 120$ (lower graph), 160 and 180 Gev (upper graph) for $m_t(m_Z) = 175$ GeV

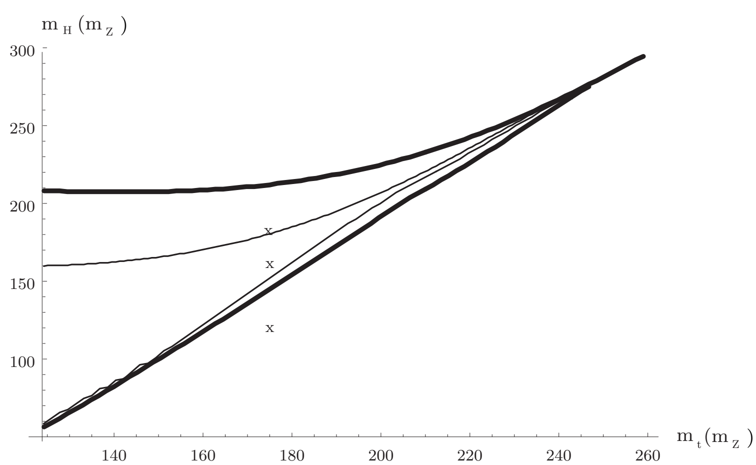

Figure 6: Two allowed domains for the Higgs mass with $\Lambda = 10^{10}$ GeV (thick lines) and $\Lambda = 10^{19}$ GeV (thin lines): slices between the upper curves ($\lambda < 1$) and the lower curves ($\lambda > 0$), with points drawn at $m_H(m_Z) = 120, 160, 180$ GeV, for $m_t(m_Z) = 175$ GeV